Positivity (2019) 23:761–778
https://doi.org/10.1007/s11117-018-0636-8

Positivity

Positive solutions for nonlinear singular superlinear elliptic equations

Yunru Bai[1] · Leszek Gasiński[1,2] · Nikolaos S. Papageorgiou[3]

Received: 27 April 2018 / Accepted: 26 November 2018 / Published online: 1 December 2018

Abstract
We consider a nonlinear nonparametric elliptic Dirichlet problem driven by the p-Laplacian and reaction containing a singular term and a $(p-1)$-superlinear perturbation. Using variational tools together with suitable truncation and comparison techniques we produce two positive, smooth, ordered solutions.

Keywords p-Laplacian · Positive solutions · Singular term · $(p-1)$-superlinear perturbation · Nonlinear regularity · Truncations

Mathematics Subject Classification 35J92 · 35J25 · 35J67

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$ and let $1 < p < +\infty$. In this paper we study the following nonlinear Dirichlet problem with a singular reaction term:

The Leszek Gasiński was supported by the National Science Center of Poland under Project No. 2015/19/B/ST1/01169.

✉ Leszek Gasiński
leszek.gasinski@up.krakow.pl

Yunru Bai
angela_baivip@163.com

Nikolaos S. Papageorgiou
npapg@math.ntua.gr

1 Faculty of Mathematics and Computer Science, Jagiellonian University, ul. Łojasiewicza 6, 30-348 Cracow, Poland

2 Department of Mathematics, Pedagogical University of Cracow, Podchorazych 2, 30-084 Cracow, Poland

3 Department of Mathematics, National Technical University, Zografou Campus, 15780 Athens, Greece

$$\begin{cases} -\Delta_p u(z) = u(z)^{-\mu} + f(z, u(z)) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \quad u > 0. \end{cases} \tag{1.1}$$

In this problem Δ_p stands for the p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}(|Du|^{p-2}Du) \quad \forall u \in W_0^{1,p}(\Omega),$$

for $1 < p < +\infty$. Also $\mu \in (0, 1)$ and $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory perturbation of the singular term (that is, for all $x \in \mathbb{R}$, $z \longmapsto f(z, x)$ is measurable and for almost all $z \in \Omega$, $x \longmapsto f(z, x)$ is continuous). We assume that $f(z, \cdot)$ is $(p-1)$-superlinear near $+\infty$ but need not satisfy the usual in such cases Ambrosetti-Rabinowitz condition.

We are looking for positive solutions and we prove the existence of at least two positive smooth solutions. Our approach is variational based on the critical point theory, together with truncation and comparison techniques.

In the past multiplicity theorems for positive solutions of singular problems were proved by Hirano et al. [20], Sun et al. [31] (semilinear problems driven by the Dirichlet Laplacian) and Giacomoni et al. [18], Kyritsi–Papageorgiou [21], Papageorgiou et al. [27], Papageorgiou–Smyrlis [28,29], Perera–Zhang [30], Zhao et al. [32]. In all aforementioned works, there is a parameter $\lambda > 0$ in the reaction term. The presence of the parameter $\lambda > 0$ permits a better control of the right-hand side nonlinearity as the parameter becomes small. In particular in [29] the authors also deal with superlinear singular problems. However, the assumptions lead to a different geometry. More precisely, in [29] the perturbation function $f(z, x)$ has a fixed sign, that is, $f(z, x) > 0$. We do not assume this here. In fact our conditions here force $f(z, \cdot)$ to be sign-changing by requiring an oscillatory behaviour near zero (see hypothesis $H(f)(i)$). Our work here complements that of [27], where the authors deal with the resonant case, that is, in [27] the perturbation $f(z, \cdot)$ is $(p-1)$-linear. The present work and [27] cover a broad class of parametric nonlinear singular Dirichlet problems. We mention also the parametric work of Aizicovici et al. [2] on singular Neumann problems. For other parametric problems see also Gasiński–Papageorgiou [7–16]. Nonparametric singular Dirichlet problems were examined by Canino–Degiovanni [4], Gasiński–Papageorgiou [6] and Mohammed [25]. In [4,25] we have existence but not multiplicity while in [6] we have also multiplicity results (the methods of proofs in all these papers are different).

2 Preliminaries and Hypotheses

Let X be a Banach space and X^* its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X). Given $\varphi \in C^1(X)$ we say that φ satisfies the Cerami condition, if the following property holds:

"Every sequence $\{u_n\}_{n\geqslant 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\geqslant 1}$ is bounded and

$$(1 + \|u_n\|)\varphi'(u_n) \longrightarrow 0 \quad \text{in } X^* \quad \text{as } n \to +\infty,$$

admits a strongly convergent subsequence."

Evidently this is a kind of compactness-type condition on the functional φ. Using the Cerami condition one can prove a deformation theorem from which follows the minimax theory of the critical values of φ. A basic result in that theory is the mountain pass theorem which we will use in the sequel.

Theorem 2.1 *If $\varphi \in C^1(X)$ satisfies the Cerami condition, $u_0, u_1 \in X$, $0 < r < \|u_1 - u_0\|$,*

$$\max\{\varphi(u_0), \varphi(u_1)\} < \inf\{\varphi(u) : \|u - u_0\| = r\} = m_r$$

and

$$c = \inf_{\gamma\in\Gamma} \max_{0\leqslant t\leqslant 1} \varphi(\gamma(t))$$

with $\Gamma = \{\gamma \in C([0,1]; X) : \gamma(0) = u_0, \gamma(1) = u_1\}$, then $c \geqslant m_r$ and c is a critical value of φ (that is, there exists $u \in X$ such that $\varphi(u) = c$ and $\varphi'(u) = 0$).

The Sobolev space $W_0^{1,p}(\Omega)$ and the Banach space $C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$ will be the two main spaces of this work. By $\|\cdot\|$ we will denote the norm of $W_0^{1,p}(\Omega)$. On account of Poincaré's inequality, we have

$$\|u\| = \|Du\|_p \quad \forall u \in W_0^{1,p}(\Omega).$$

The Banach space $C_0^1(\overline{\Omega})$ is an ordered Banach space with positive (order) cone

$$C_+ = \{u \in C_0^1(\overline{\Omega}) : u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega, \quad \frac{\partial u}{\partial n}|_{\partial\Omega} < 0\right\}.$$

Here $\frac{\partial u}{\partial n}$ denotes the normal derivative of u defined by

$$\frac{\partial u}{\partial n} = (Du, n)_{\mathbb{R}^N},$$

with n being the outward unit normal on $\partial\Omega$.

Let $A: W_0^{1,p}(\Omega) \longrightarrow W_0^{1,p}(\Omega)^* = W^{-1,p'}(\Omega)$ $(\frac{1}{p} + \frac{1}{p'} = 1)$ be the nonlinear map defined by

$$\langle A(u), h\rangle = \int_\Omega |Du|^{p-2}(Du, Dh)_{\mathbb{R}^N}\, dz \quad \forall u, h \in W_0^{1,p}(\Omega).$$

In the next proposition, we recall the main properties of this map (see Motreanu et al. [26, p. 40]).

Proposition 2.2 *The map $A\colon W_0^{1,p}(\Omega) \longrightarrow W^{-1,p'}(\Omega)$ is bounded (that is, maps bounded sets to bounded sets), continuous, strictly monotone (thus maximal monotone) and of type $(S)_+$, that is,*

"if $u_n \xrightarrow{w} u$ in $W_0^{1,p}(\Omega)$ and $\limsup_{n\to+\infty}\langle A(u_n), u_n - u\rangle \leqslant 0$, then $u_n \longrightarrow u$ in $W_0^{1,p}(\Omega)$."

By p^* we denote the critical Sobolev exponent corresponding to p, i.e.,

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if} \quad p < n, \\ +\infty & \text{if} \quad N \leqslant p. \end{cases}$$

The hypotheses on the perturbation term f are the following:
$\underline{H(f)}$: $f\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for a.a. $z \in \Omega$ and

(i) there exist $a \in L^\infty(\Omega)$ and $r \in (p, p^*)$ such that

$$|f(z,x)| \leqslant a(z)(1 + x^{r-1}) \quad \text{for a.a. } z \in \Omega, \text{ all } x \geqslant 0$$

and there exists $w \in C^1(\overline{\Omega})$ such that

$$w(z) \geqslant \widehat{c} > 0 \text{ for all } z \in \overline{\Omega}, \quad \Delta_p w \in L^\infty(\Omega), \quad \Delta_p w \leqslant 0 \text{ for a.a. } z \in \Omega$$

and for every compact set $K \subseteq \Omega$, there exists $c_K > 0$ such that

$$w(z)^{-\mu} + f(z, w(z)) \leqslant -c_K < 0 \quad \text{for a.a. } z \in K;$$

(ii) if $F(z,x) = \int_0^x f(z,s)\,ds$ and for every $\lambda > 0$ we define

$$\xi_\lambda(z,x) = \left(\frac{p}{1-\mu} - 1\right)x^{1-\mu} + \lambda(f(z,x)x - pF(z,x)),$$

then

$$\lim_{x\to+\infty} \frac{F(z,x)}{x^p} = +\infty \quad \text{uniformly for a.a. } z \in \Omega,$$

and there exists $\beta_\lambda \in L^1(\Omega)$, $\beta_\lambda(z) \geqslant 0$ for a.a. $z \in \Omega$ such that

$$\xi_\lambda(z,x) \leqslant \xi_\lambda(z,y) + \beta_\lambda(z) \quad \text{for a.a. } z \in \Omega, \text{ all } 0 \leqslant x \leqslant y;$$

(iii) there exists $\delta \in (0, \widehat{c}]$ such that

$$f(z,x) \geqslant 0 \quad \text{for a.a. } z \in \Omega, \text{ all} 0 \leqslant x \leqslant \delta;$$

(iv) for every $\varrho > 0$, there exists $\widehat{\xi}_\varrho > 0$ such that for a.a. $z \in \Omega$ the function

$$x \longmapsto f(z, x) + \widehat{\xi}_\varrho x^{p-1}$$

is nondecreasing on $[0, \varrho]$.

Remark 2.3 Since we look for positive solutions and the above hypotheses concern the positive semiaxes $\mathbb{R}_+ = [0, +\infty)$, without any loss of generality, we assume that

$$f(z, x) = 0 \quad \text{for a.a.} \quad z \in \Omega, \ \text{all } x \leqslant 0. \tag{2.1}$$

Hypothesis $H(f)(ii)$ implies that for a.a. $z \in \Omega$, $f(z, \cdot)$ is $(p-1)$-superlinear, that is,

$$\lim_{x \to +\infty} \frac{f(z, x)}{x^{p-1}} = +\infty \quad \text{uniformly for a.a. } z \in \Omega.$$

We stress that for the superlinearity of $f(z, \cdot)$ we do not use the Ambrosetti–Rabinowitz condition which says that there exist $r > p$ and $M > 0$ such that

$$0 < rF(z, x) \leqslant f(z, x)x \quad \text{for a.a. } z \in \Omega, \ \text{all } x \geqslant M, \quad \operatorname*{ess\,inf}_{\Omega} F(\cdot, M) > 0.$$

This condition implies that $f(z, \cdot)$ has at least x^{r-1}-growth near $+\infty$, that is

$$c_0 x^{r-1} \leqslant f(z, x) \quad \text{for a.a. } z \in \Omega, \ \text{all } x \geqslant M,$$

for some $c_0 > 0$. This excludes from consideration $(p-1)$-superlinear nonlinearities with "slower" growth near $+\infty$ (see Example 2.4). Here we replace the Ambrosetti–Rabinowitz condition with a quasimonotonicity condition on $\xi(z, \cdot)$ (see hypothesis $H(f)(ii)$), which incorporates in our framework more superlinear nonlinearities. Hypothesis $H(f)(ii)$ is a slight generalization of a condition used by Li–Yang [23]. It is satisfied, if there is $M > 0$ such that for a.a. $z \in \Omega$, the function $x \longmapsto \frac{f(z,x)}{x^{p-1}}$ is nondecreasing on $[M, +\infty)$ and this in turn is equivalent to saying that for a.a. $z \in \Omega$, $\xi(z, \cdot)$ is nondecreasing on $[M, +\infty)$. For details see Li–Yang [23]. Hypotheses $H(f)(i)$ and (iii) imply that for a.a. $z \in \Omega$, $f(z, \cdot)$ exhibits a kind of oscillatory behaviour near zero. In hypothesis $H(f)(i)$, the condition $\Delta_p w(z) \leqslant 0$ for a.a. $z \in \Omega$, implies that

$$0 \leqslant \int_\Omega |Dw|^{p-2}(Dw, Dh)_{\mathbb{R}^N}\, dz \quad \text{for all } h \in W_0^{1,p}(\Omega), \quad h(z) \geqslant 0 \quad \text{for a.a. } z \in \Omega.$$

Evidently the condition with $w(\cdot)$ in hypothesis $H(f)(i)$ is satisfied if $w(z) \equiv c_+ > 0$ for all $z \in \overline{\Omega}$ and $\operatorname*{ess\,inf}_{\Omega} f(\cdot, c_+) < -\frac{1}{c_+^{\mu}}$. So, hypotheses $H(f)(i)$ and (ii) dictate an oscillatory behaviour for $f(z, \cdot)$ near zero.

Example 2.4 The following function satisfies hypotheses $H(f)$. For the sake of simplicity we drop the z-dependence:

$$f(x)=\begin{cases} x^{p-1}-cx^{r-1} & \text{if} \quad 0 \leqslant x \leqslant 1, \\ x^{p-1}\ln x+(1-c)x^{q-1} & \text{if} \quad 1 < x, \end{cases}$$

with $1 < q < p < r < +\infty$ and $c > 2$ [see (2.1)]. Note that f although $(p-1)$-superlinear, it fails to satisfy the Ambrosetti–Rabinowitz condition.

Finally let us fix our notation. If $x \in \mathbb{R}$, we set $x^{\pm} = \max\{\pm x, 0\}$. Then given $u \in W_0^{1,p}(\Omega)$ we define $u^{\pm}(\cdot) = u(\cdot)^{\pm}$ and we have

$$u^{\pm} \in W_0^{1,p}(\Omega), \quad u = u^{+} - u^{-}, \quad |u| = u^{+} + u^{-}.$$

Set $\widehat{C}_{+} = \{u \in C^1(\overline{\Omega}) : \ u|_{\overline{\Omega}} \geqslant 0, \ \frac{\partial u}{\partial n} \leqslant 0 \text{ on } \partial\Omega \cap u^{-1}(0)\}$. We also mention that when we want to emphasize the domain D on which the cones C_+ and $\operatorname{int} C_+$ are considered, we write $C_+(D)$ and $\operatorname{int} C_+(D)$.

Moreover, by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$ and if $\varphi \in C^1(X)$, then

$$K_{\varphi} = \{u \in X : \ \varphi'(u) = 0\}$$

(the "critical set" of φ).

3 Positive Solutions

In this section we prove the existence of two positive smooth solution for problem (1.1).

Proposition 3.1 *If hypotheses $H(f)(i)$ and (iii) hold, then there exists $\underline{u} \in \operatorname{int} C_+$ such that*

$$\begin{cases} -\Delta_p \underline{u}(z) \leqslant \underline{u}(z)^{-\mu} + f(z, \underline{u}(z)) & \textit{for a.a. } z \in \Omega \\ \underline{u} \leqslant w \end{cases}$$

Proof We consider the following auxiliary singular Dirichlet problem

$$\begin{cases} -\Delta_p u(z) = u(z)^{-\mu} & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \quad u > 0. \end{cases}$$

From Proposition 5 of Papageorgiou–Smyrlis [29], we know that this problem has a unique positive solution $\widetilde{u} \in \operatorname{int} C_+$.

With $\widehat{c} > 0$ and $\delta > 0$ as postulated by hypotheses $H(f)(i)$ and (iii) respectively, we choose

$$t \in \left(0, \min\left\{1, \frac{\widehat{c}}{\|\widetilde{u}\|_\infty}, \frac{\delta}{\|\widetilde{u}\|_\infty}\right\}\right).$$

We set $\underline{u} = t\widetilde{u} \in \operatorname{int} C_+$. We have

$$\begin{aligned} -\Delta_p \underline{u}(z) &= t^{p-1}(-\Delta_p \widetilde{u}(z)) \;=\; t^{p-1}\widetilde{u}(z)^{-\mu} \\ &\leqslant \underline{u}(z)^{-\mu} \;\leqslant\; \underline{u}(z)^{-\mu} + f(z, \underline{u}(z)) \quad \text{for a.a. } z \in \Omega \end{aligned}$$

(recall that $t \leqslant 1$ and see hypothesis $H(f)(iii)$ and Papageorgiou–Smyrlis [29]). Moreover, we have $\underline{u} \leqslant w$. $\square$

Using $\underline{u} \in \operatorname{int} C_+$, from Proposition 3.1 and $w \in C^1(\overline{\Omega})$ from hypothesis $H(f)(i)$, we introduce the following truncation of $f(z, \cdot)$:

$$\widehat{g}(z, x) = \begin{cases} \underline{u}(z)^{-\mu} + f(z, \underline{u}(z)) & \text{if} \quad x < \underline{u}(z), \\ x^{-\mu} + f(z, x) & \text{if} \quad \underline{u}(z) \leqslant x \leqslant w(z), \\ w(z)^{-\mu} + f(z, w(z)) & \text{if} \quad w(z) < x. \end{cases} \tag{3.1}$$

Given $y, v \in W^{1,p}(\Omega)$, $y \leqslant v$, we define

$$[y, v] = \{u \in W_0^{1,p}(\Omega) : \; y(z) \leqslant u(z) \leqslant v(z) \text{ for a.a. } z \in \Omega\}.$$

Also by $\operatorname{int}_{C_0^1(\overline{\Omega})}[y, v]$ we denote the interior in the $C_0^1(\overline{\Omega})$-norm topology of $[y, v] \cap C_0^1(\overline{\Omega})$.

Proposition 3.2 *If hypotheses $H(f)(i)$ and (iii) hold, then problem (1.1) admits a solution $u_0 \in [\underline{u}, w] \cap C_0^1(\overline{\Omega})$.*

Proof Let

$$\widehat{G}(z, x) = \int_0^x \widehat{g}(z, s)\, ds$$

and consider the functional $\widehat{\varphi} \colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\widehat{\varphi}(u) = \frac{1}{p}\|Du\|_p^p - \int_\Omega \widehat{G}(z, u)\, dz \qquad \forall u \in W_0^{1,p}(\Omega).$$

Proposition 3 of Papageorgiou–Smyrlis [29] implies that $\widehat{\varphi} \in C^1(W_0^{1,p}(\Omega))$ and we have

$$\langle \widehat{\varphi}'(u), h\rangle = \langle A(u), h\rangle - \int_\Omega \widehat{g}(z, u) h\, dz \quad \forall h \in W_0^{1,p}(\Omega).$$

From (3.1) it is clear that $\widehat{\varphi}$ is coercive. Also, the Sobolev embedding theorem implies that $\widehat{\varphi}$ is sequentially weakly lower semicontinuous. So, by the Weierstrass–Tonelli theorem, we can find $u_0 \in W_0^{1,p}(\Omega)$ such that

$$\widehat{\varphi}(u_0) = \inf_{u \in W_0^{1,p}(\Omega)} \widehat{\varphi}(u),$$

so $\widehat{\varphi}'(u_0) = 0$, hence

$$\langle A(u_0), h\rangle = \int_\Omega \widehat{g}(z, u_0)h \quad dz \quad \forall h \in W_0^{1,p}(\Omega). \tag{3.2}$$

In (3.2) first we choose $h = (\underline{u} - u_0)^+ \in W_0^{1,p}(\Omega)$. We have

$$\begin{aligned}\langle A(u_0), (\underline{u} - u_0)^+\rangle \ &= \int_\Omega \widehat{g}(z, u_0)(\underline{u} - u_0)^+ \, dz \\ = \int_\Omega (\underline{u}^{-\mu} + f(z, \underline{u}))(\underline{u} - u_0)^+ \, dz \ &\geqslant \ \langle A(\underline{u}), (\underline{u} - u_0)^+\rangle\end{aligned}$$

[see (3.1)] and Proposition 3.1), so

$$\langle A(\underline{u}) - A(u_0), (\underline{u} - u_0)^+\rangle \leqslant 0,$$

hence $\underline{u} \leqslant u_0$.

Next in (3.2) we choose $h = (u_0 - w)^+ \in W_0^{1,p}(\Omega)$ (see hypothesis $H(f)(i)$). Then we have

$$\begin{aligned}\langle A(u_0), (u_0 - w)^+\rangle \ &= \int_\Omega \widehat{g}(z, u_0)(u_0 - w)^+ \, dz \\ = \int_\Omega (w^{-\mu} + f(z, w))(u_0 - w)^+ \, dz \ &\leqslant \ \langle A(w), (u_0 - w)^+\rangle\end{aligned}$$

[see (3.1)] and hypothesis $H(f)(i)$), so

$$\langle A(u_0) - A(w), (u_0 - w)^+\rangle \leqslant 0,$$

hence $u_0 \leqslant w$. So, we have proved that

$$u_0 \in [\underline{u}, w]. \tag{3.3}$$

From (3.1), (3.2) and (3.3), we have

$$\langle A(u_0), h\rangle = \int_\Omega (u_0^{-\mu} + f(z, u_0))h \, dz \qquad \forall h \in W_0^{1,p}(\Omega). \tag{3.4}$$

Let $d(z) = d(z, \partial\Omega)$ for $z \in \overline{\Omega}$ (the distance from the boundary $\partial\Omega$). Then Lemma 14.16 of Gilbarg-Trudinger [19, p. 355] implies that there exists $\delta_0 > 0$ such that

$$d \in \operatorname{int} \widehat{C}_+(\Omega_{\delta_0}),$$

where $\Omega_{\delta_0} = \{z \in \Omega : \ d(z) = d(z, \partial\Omega) < \delta_0\}$. Let $D = \overline{\Omega} \setminus \Omega_{\delta_0}$ and consider the ordered Banach space $C(D)$ with positive (order) cone $C(D)_+$. Since $u_0(z) \geqslant \widetilde{c} > 0$

for all $z \in D$, it follows that

$$d \in \operatorname{int} C(D)_+.$$

Recall that $\underline{u} \in \operatorname{int} C_+$ (see Proposition 3.1). So, on account of Proposition 2.1 of Marano–Papageorgiou [24], we can find $0 < c_1 < c_2$ such that

$$c_1 d \leqslant \underline{u} \leqslant c_2 d. \tag{3.5}$$

For all $h \in W_0^{1,p}(\Omega)$ we have

$$\left| \int_\Omega u_0^{-\mu} h \, dz \right| \leqslant \frac{1}{c_1^\mu} \int_\Omega d^{1-\mu} \frac{|h|}{d} dz \leqslant c_3 \int_\Omega \frac{|h|}{d} \, dz \leqslant c_4 \|h\|$$

for some $c_3, c_4 > 0$ (since $\Omega \subseteq \mathbb{R}^N$ is bounded, $\mu \in (0, 1)$ and using Hardy's inequality; see Brézis [3, p. 313]).

Therefore from (3.4) and since $u_0^{-\mu} \in L^1(\Omega)$ (see Lazer-McKenna [22, Lemma]), it follows that

$$\begin{cases} -\Delta_p u_0(z) = u_0(z)^{-\mu} + f(z, u_0(z)) & \text{in } \Omega, \\ u_0|_{\partial\Omega} = 0. \end{cases}$$

Invoking Theorem B.1 of Giacomoni-Schindler-Takáč [18], we have that $u_0 \in \operatorname{int} C_+$. Therefore finally we can say that $u_0 \in [\underline{u}, w] \cap C_0^1(\overline{\Omega})$. □

If we strengthen the conditions on the perturbation term $f(z, x)$ we can improve the condition of Proposition 3.2.

Proposition 3.3 *If hypotheses $H(f)(i)$, (iii) and (iv) hold, then*

$$u_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})}[\underline{u}, w].$$

Proof From Proposition 3.2 we already know that

$$u_0 \in [\underline{u}, w] \cap C_0^1(\overline{\Omega}).$$

Let $\varrho = \|w\|_\infty$ and let $\widehat{\xi}_\varrho > 0$ be as postulated by hypothesis $H(f)(iv)$. We have

$$\begin{aligned} -\Delta_p u_0(z) - u_0(z)^{-\mu} + \widehat{\xi}_\varrho u_0(z)^{p-1} &= f(z, u_0(z)) + \widehat{\xi}_\varrho u_0(z)^{p-1} \\ &\geqslant f(z, \underline{u}(z)) + \widehat{\xi}_\varrho \underline{u}(z)^{p-1} > \widehat{\xi}_\varrho \underline{u}(z)^{p-1} \\ &\geqslant -\Delta_p \underline{u}(z) - \underline{u}(z)^{-\mu} + \widehat{\xi}_\varrho \underline{u}(z)^{p-1} \quad \text{for a.a. } z \in \Omega \end{aligned} \tag{3.6}$$

[see (3.3)], hypotheses $H(f)(iv)$, (iii) and Proposition 3.1). Then (3.6) and Proposition 4 of Papageorgiou–Smyrlis [29], imply that

$$u_0 - \underline{u} \in \operatorname{int} C_+.$$

Let $D_0 = \{z \in \Omega : u_0(z) = w(z)\}$. The hypothesis on the function w (see hypothesis $H(f)(i)$), implies that $D_0 \subseteq \Omega$ is compact. So, we can find an open set $U \subseteq \Omega$ with C^2-boundary ∂U such that

$$D_0 \subseteq U \subseteq \overline{U} \subseteq \Omega.$$

We have

$$\begin{aligned} -\Delta_p w(z) - w(z)^{-\mu} + \widehat{\xi}_\varrho w(z)^{p-1} &\geqslant c_{\overline{U}} + f(z.w(z)) + \widehat{\xi}_\varrho w(z)^{p-1} \\ &\geqslant f(z, w(z)) + \widehat{\xi}_\varrho w(z)^{p-1} \geqslant f(z, u_0(z)) + \widehat{\xi}_\varrho u_0(z)^{p-1} \\ &= -\Delta_p u_0(z) - u_0(z)^{-\mu} + \widehat{\xi}_\varrho u_0(z)^{p-1} \quad \text{for a.a. } z \in U \end{aligned}$$

[see (3.3) and hypotheses $H(f)(i)$ and (iv)]. Then Proposition 5 of Papageorgiou–Smyrlis [29] (the "singular" strong comparison principle) implies that

$$w - u_0 \in \operatorname{int} C_+(\overline{U}).$$

Since $D_0 \subseteq U$, it follows that $D_0 = \emptyset$ and so we have

$$u_0(z) < w(z) \qquad \forall z \in \overline{\Omega}.$$

Therefore, we conclude that $u_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})}[\underline{u}, w]$. □

Next we produce a second positive solution for problem (1.1).

Proposition 3.4 *If hypotheses $H(f)$ hold, then problem (1.1) admits a second positive solution $\widehat{u} \in \operatorname{int} C_+$.*

Proof We introduce the following truncation of the reaction term in problem (1.1):

$$e(z, x) = \begin{cases} \underline{u}(z)^{-\mu} + f(z, \underline{u}(z)) & \text{if} \quad x \leqslant \underline{u}(z) \\ x^{-\mu} + f(z, x) & \text{if} \quad \underline{u}(z) < x. \end{cases} \tag{3.7}$$

Clearly this is a Carathéodory function. We set $E(z, x) = \int_0^x e(z, s)\, ds$ and consider the functional $\varphi_* : W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\varphi_*(u) = \frac{1}{p}\|Du\|_p^p - \int_\Omega E(z, u) \quad dz \quad \forall u \in W_0^{1,p}(\Omega).$$

We know that $\varphi_* \in C^1(W_0^{1,p}(\Omega))$ (see Papageorgiou–Smyrlis [29, Proposition 3]).
Claim: φ_* satisfies the Cerami condition.

Let $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ be a sequence such that

$$|\varphi_*(u_n)| \leqslant M_1 \quad \forall n \in \mathbb{N}, \quad \text{for some } M_1 > 0, \tag{3.8}$$
$$(1 + \|u_n\|)\varphi_*'(u_n) \longrightarrow 0 \quad \text{in } W^{-1,p'}(\Omega) \quad \text{as } n \to +\infty. \tag{3.9}$$

From (3.9) we have

$$\left|\langle A(u_n), h\rangle - \int_\Omega e(z, u_n)h\,dz\right| \leqslant \frac{\varepsilon_n \|h\|}{1 + \|u_n\|} \quad \forall h \in W_0^{1,p}(\Omega), \tag{3.10}$$

with $\varepsilon_n \to 0^+$. In (3.10) we choose $h = -u_n^- \in W_0^{1,p}(\Omega)$. Then

$$\|Du_n^-\|_p^p - \int_\Omega (\underline{u}^{-\mu} + f(z, \underline{u}))(-u_n^-)\,dz \leqslant \varepsilon_n \quad \forall n \in \mathbb{N}$$

[see (3.7)], so

$$\|Du_n^-\|_p^p \leqslant c_5(1 + \|u_n^-\|) \quad \forall n \in \mathbb{N},$$

for some $c_5 > 0$, thus

$$\text{the sequence } \{u_n^-\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.} \tag{3.11}$$

We use (3.11) in (3.8) and we have

$$\left|\|Du_n^+\|_p^p - \int_\Omega pE(z, u_n^+)\,dz\right| \leqslant M_2 \quad \forall n \in \mathbb{N}, \tag{3.12}$$

for some $M_2 > 0$. Also, if in (3.10) we choose $h = u_n^+ \in W_0^{1,p}(\Omega)$, then

$$-\|Du_n^+\|_p^p + \int_\Omega e(z, u_n^+)u_n^+\,dz \leqslant \varepsilon_n \quad \forall n \in \mathbb{N}. \tag{3.13}$$

We add (3.12) and (3.13) and obtain

$$\int_\Omega (e(z, u_n^+)u_n^+ - pE(z, u_n^+))\,dz \leqslant M_3 \quad \forall n \in \mathbb{N},$$

for some $M_3 > 0$, so

$$\int_\Omega (f(z, u_n^+)u_n^+ - pF(z, u_n^+)\,dz \leqslant M_4 \quad \forall n \in \mathbb{N},$$

for some $M_4 > 0$ [see (3.7)], thus

$$\int_\Omega \xi(z, u_n^+)\,dz \leqslant M_4 \quad \forall n \in \mathbb{N}. \tag{3.14}$$

Suppose that the sequence $\{u_n^+\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ is not bounded. By passing to a subsequence if necessary, we may assume that

$$\|u_n^+\| \longrightarrow +\infty.$$

Let $y_n = \frac{u_n^+}{\|u_n^+\|}$ for $n \in \mathbb{N}$. Then $\|y_n\| = 1$, $y_n \geqslant 0$ for all $n \in \mathbb{N}$. So, passing to a next subsequence if necessary, we may assume that

$$y_n \xrightarrow{w} y \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and} \quad y_n \longrightarrow y \quad \text{in } L^p(\Omega), \tag{3.15}$$

with $y \geqslant 0$.

First assume that $y \neq 0$. Let $\Omega_+ = \{z \in \Omega : y(z) > 0\}$. We have $|\Omega_+|_N > 0$ [see (3.15)] and

$$u_n^+(z) \longrightarrow +\infty \quad \text{for a.a. } z \in \Omega_+.$$

Hypothesis $H(f)(ii)$ implies that

$$\frac{F(z, u_n^+(z))}{\|u_n^+\|^p} = \frac{F(z, u_n^+(z))}{u_n^+(z)^p} y_n(z)^p \longrightarrow +\infty \quad \text{for a.a. } z \in \Omega_+. \tag{3.16}$$

From (3.16) and Fatou's lemma we have

$$\int_{\Omega_+} \frac{F(z, u_n^+)}{\|u_n^+\|^p}\, dz \longrightarrow +\infty \quad \text{as } n \to +\infty. \tag{3.17}$$

On the other hand hypothesis $H(f)(ii)$ implies that we can find $M_5 > 0$ such that

$$F(z, x) \geqslant 0 \quad \text{for a.a. } z \in \Omega, \quad \text{all } x \geqslant M_5.$$

It follows that

$$\int_{\Omega \setminus \Omega_+} \frac{F(z, u_n^+)}{\|u_n^+\|^p}\, dz \geqslant -c_6 \quad \forall n \in \mathbb{N}, \tag{3.18}$$

for some $c_6 > 0$. From (3.17) and (3.18) we infer that

$$\int_{\Omega} \frac{F(z, u_n^+)}{\|u_n^+\|^p}\, dz \longrightarrow +\infty \quad \text{as } n \to +\infty. \tag{3.19}$$

On the other hand, from (3.12) we have

$$\int_{\Omega} \frac{pE(z, u_n^+)}{\|u_n^+\|^p}\, dz \leqslant c_7(1 + \|Dy_n^+\|_p^p) \quad \forall n \in \mathbb{N},$$

for some $c_7 > 0$, so

$$\int_{\Omega} \frac{pF(z, u_n^+)}{\|u_n^+\|^p}\, dz \leqslant c_8 \quad \forall n \in \mathbb{N}, \tag{3.20}$$

for some $c_8 > 0$. Comparing (3.19) and (3.20), we have a contradiction. This proves the Claim when $y \neq 0$.

Next assume that $y = 0$. For $k > 0$, let $v_n = (kp)^{\frac{1}{p}} y_n$ for $n \in \mathbb{N}$. Then from (3.15) we have

$$v_n \xrightarrow{w} 0 \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and} \quad v_n \longrightarrow 0 \quad \text{in } L^p(\Omega). \tag{3.21}$$

We can find $n_0 \in \mathbb{N}$ such that

$$0 < (kp)^{\frac{1}{p}} \frac{1}{\|u_n^+\|} \leqslant 1 \quad \forall n \geqslant n_0. \tag{3.22}$$

Let $t_n \in [0, 1]$ be such that

$$\varphi_*(t_n u_n^+) = \max_{0 \leqslant t \leqslant 1} \varphi_*(t u_n^+) \quad \forall n \in \mathbb{N}. \tag{3.23}$$

From (3.21) and Krasonoselskii's theorem (see Gasiński–Papageorgiou [5, Theorem 3.4.4, p.407]), we have

$$\int_\Omega E(z, v_n)\, dz \longrightarrow 0 \quad \text{as } n \to +\infty. \tag{3.24}$$

From (3.22) and (3.23), we have

$$\begin{aligned} \varphi_*(t_n u_n^+) \geqslant \varphi_*(v_n) &= \frac{1}{p}\|Dv_n\|_p^p - \int_\Omega E(z, v_n)\, dz \\ &\geqslant k - \int_\Omega E(z, v_n)\, dz \geqslant \frac{k}{2} \quad \forall n \geqslant n_1 \geqslant n_0 \end{aligned}$$

[see (3.24)]. But $k > 0$ is arbitrary. So, we infer that

$$\varphi_*(t_n u_n^+) \longrightarrow +\infty \quad \text{as } n \to +\infty. \tag{3.25}$$

We know that

$$\varphi_*(0) = 0 \quad \text{and} \quad \varphi_*(u_n^+) \leqslant M_6 \quad \forall n \in \mathbb{N}, \tag{3.26}$$

for some $M_6 > 0$ [see (3.8) and (3.11)]. From (3.25), (3.26) and (3.23) it follows that

$$t_n \in (0, 1) \quad \forall n \geqslant n_2.$$

Then we have

$$0 = \frac{d}{dt}\varphi_*(t u_n^+)|_{t=t_n} = \langle \varphi_*'(t_n u_n^+), u_n^+ \rangle$$

(by the chain rule), so

$$\|D(t_n u_n^+)\|_p^p = \int_\Omega e(z, t_n u_n^+)(t_n u_n^+)\, dz \quad \forall n \geqslant n_2. \tag{3.27}$$

We have

$$\begin{aligned}
&\int_\Omega e(z, t_n u_n^+)(t_n u_n^+)\,dz\\
&= \int_{\{0\leqslant t_n u_n^+\leqslant \underline{u}\}} (\underline{u}^{-\mu} + f(z, \underline{u}))(t_n u_n^+)\,dz\\
&\quad + \int_{\{\underline{u}\leqslant t_n u_n^+\}} ((t_n u_n)^{-\mu} + f(z, t_n u_n^+))(t_n u_n^+)\,dz\\
&\leqslant \int_{\{0\leqslant t_n u_n^+\leqslant \underline{u}\}} \underline{u}^{-\mu}(t_n u_n^+)\,dz + \int_{\{\underline{u}\leqslant t_n u_n^+\}} (t_n u_n^+)^{1-\mu}\,dz\\
&\quad + \int_{\{0\leqslant t_n u_n^+\leqslant \underline{u}\}} f(z, \underline{u})(t_n u_n^+)\,dz + \int_{\{\underline{u}\leqslant t_n u_n^+\}} \xi(z, u_n^+)\,dz\\
&\quad + \int_{\{\underline{u}\leqslant t_n u_n^+\}} pF(z, t_n u_n^+)\,dz + \|\beta\|_1
\end{aligned} \tag{3.28}$$

[see (3.7) and hypothesis $H(f)(ii)$].

We use (3.28) in (3.27) and recall that $\underline{u}(z)^{-\mu} + f(z, \underline{u}(z)) \geqslant 0$ for a.a. $z \in \Omega$ (see hypothesis $H(f)(iii)$). We have

$$\begin{aligned}
&\|D(t_n u_n^+)\|_p^p - p\int_{\{0\leqslant t_n u_n^+\leqslant \underline{u}\}} (\underline{u}^{-\mu} + f(z, \underline{u}))(t_n u_n^+)\,dz\\
&\quad - \frac{p}{1-\mu}\int_{\{\underline{u}\leqslant t_n u_n^+\}} (t_n u_n^+)^{1-\mu}\,dz - \int_{\{\underline{u}\leqslant t_n u_n^+\}} pF(z, t_n u_n^+)\,dz\\
&\leqslant \int_\Omega \xi(z, u_n^+)\,dz + \|\beta\|_1
\end{aligned}$$

(see hypothesis $H(f)(ii)$), so

$$p\varphi_*(t_n u_n^+) \leqslant \int_\Omega \xi(z, u_n^+)\,dz + c_9 \leqslant c_{10} \quad \forall n \in \mathbb{N}. \tag{3.29}$$

for some $c_9, c_{10} > \|\beta\|_1$. Comparing (3.25) and (3.29), we have a contradiction.

So, we have proved that

$$\text{the sequence } \{u_n^+\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.} \tag{3.30}$$

From (3.11) and (3.30) we infer that

$$\text{the sequence } \{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.}$$

So, passing to a subsequence if necessary, we may assume that

$$u_n \xrightarrow{w} u \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \longrightarrow u \quad \text{in } L^p(\Omega). \tag{3.31}$$

In (3.10) we choose $h = u_n - u \in W_0^{1,p}(\Omega)$. We have

$$\langle A(u_n), u_n - u\rangle - \int_\Omega e(z, u_n)(u_n - u)\, dz \leqslant \varepsilon_n' \quad \forall n \in \mathbb{N}, \tag{3.32}$$

with $\varepsilon_n' \to 0^+$. Note that

$$\begin{aligned}\int_\Omega & e(z, u_n)(u_n - u)\, dz \\ &= \int_{\{u_n \leqslant \underline{u}\}} (\underline{u}^{-\mu} + f(z, \underline{u}))(u_n - n)\, dz \\ &\quad + \int_{\{\underline{u} < u_n\}} (u_n^{-\mu} + f(z, u_n))(u_n - u)\, dz \quad \forall n \in \mathbb{N}\end{aligned} \tag{3.33}$$

[see (3.7)]. Recall that $\underline{u} \in \operatorname{int} C_+$. Hence we can find $c_{11} > 0$ such that

$$\widehat{u}_1 \leqslant c_{11}\underline{u}^{p'}$$

(see Proposition 2.1 of Marano–Papageorgiou [24]), so

$$\widehat{u}^{\frac{1}{p'}} \leqslant c_{11}^{\frac{1}{p'}}\underline{u},$$

thus

$$c_{12}\underline{u}^{-\mu} \leqslant \widehat{u}_1^{-\frac{\mu}{p'}},$$

for some $c_{12} > 0$. From Lazer–McKenna [22, Lemma], we know that $\widehat{u}_1^{-\frac{\mu}{p'}} \in L^{p'}(\Omega)$, so $c_{12}\underline{u}^{-\mu} \in L^{p'}(\Omega)$. Therefore, we have

$$\int_{\{u_n \leqslant \underline{u}\}} (\underline{u}^{-\mu} + f(z, \underline{u}))(u_n - u)\, dz \longrightarrow 0 \quad \text{as } n \to +\infty \tag{3.34}$$

[see (3.31)]. Similarly, we have

$$\int_{\{\underline{u} < u_n\}} (u_n^{-\mu} + f(z, u_n))(u_n - u)\, dz \longrightarrow 0 \quad \text{as } n \to +\infty. \tag{3.35}$$

We return to (3.32), pass to the limit as $n \to +\infty$ and use (3.33), (3.34), (3.35). We obtain

$$\lim_{n \to +\infty} \langle A(u_n), u_n - u\rangle = 0,$$

so $u_n \to u$ in $W_0^{1,p}(\Omega)$ (see Proposition 2.2) and thus φ_* satisfies the Cerami condition. This proves the Claim.

From (3.1) and (3.7) we see that

$$\widehat{\varphi}|_{[\underline{u},w]} = \varphi_*|_{[\underline{u},w]} \tag{3.36}$$

(here $\widehat{\varphi}$ is as in the proof of Proposition 3.2). From the proof of Proposition 3.2, we know that

$$u_0 \in \operatorname{int} C_+ \text{ is a minimizer of } \widehat{\varphi}, \tag{3.37}$$

while from Proposition 3.3, we know that

$$u_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})}[\underline{u}, w]. \tag{3.38}$$

Then (3.36), (3.37) and (3.38) imply that

$$u_0 \text{ is a local } C_0^1(\overline{\Omega})\text{-minimizer of } \varphi_*,$$

thus

$$u_0 \text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \varphi_* \tag{3.39}$$

(see Theorem 1.1 of Giacomoni–Saoudi [17]). Using (3.7) we can easily see that

$$K_{\varphi_*} \subseteq \{u \in C_0^1(\overline{\Omega}) : \ \underline{u}(z) \leqslant u(z) \quad \text{for all } z \in \overline{\Omega}\}. \tag{3.40}$$

Therefore we may assume that K_{φ_*} is finite or otherwise we already have an infinity of positive smooth solutions of (1.1) [see (3.7)] all bigger than u_0 and so we are done. The finiteness of K_{φ_*} and (3.39) imply that we can find $\varrho \in (0, 1)$ small such that

$$\varphi_*(u_0) < \inf\{\varphi_*(u) : \ \|u - u_0\| = \varrho\} = m_* \tag{3.41}$$

(see Aizicovici et al. [1, proof of Proposition 29]). Hypothesis $H(f)(ii)$ implies that if $u \in \operatorname{int} C_+$, then

$$\varphi_*(tu) \longrightarrow -\infty \quad \text{as } t \to +\infty. \tag{3.42}$$

Then (3.41), (3.42) and the Claim permit the use of the mountain pass theorem (see Theorem 2.1). So, we can find $\widehat{u} \in W_0^{1,p}(\Omega)$ such that

$$\widehat{u} \in K_{\varphi_*} \quad \text{and} \quad m_* \leqslant \varphi_*(\widehat{u}). \tag{3.43}$$

From (3.40), (3.41), (3.43) and (3.7) we conclude that $\widehat{u} \in \operatorname{int} C_+$, $\widehat{u} \neq u_0$, $\widehat{u}$ is a positive solution of (1.1) and $\widehat{u} \geqslant u_0$. $\square$

We can state the following multiplicity theorem for problem (1.1).

Theorem 3.5 *If hypotheses $H(f)$ hold, then problem (1.1) has two positive smooth solutions*

$$u_0, \widehat{u} \in \operatorname{int} C_+, \ \ \widehat{u} - u_0 \in C_+ \setminus \{0\}.$$

Acknowledgements The authors wish to thank a knowledgable referee for his/her corrections and helpful remarks.

References

1. Aizicovici, S., Papageorgiou, N.S., Staicu, V.: Degree Theory for Operators of Monotone Type and Nonlinear Elliptic Equations with Inequality Constraints. Memoirs of the American Mathematical Society. **196**(195), 70 (2008)
2. Aizicovici, S., Papageorgiou, N.S., Staicu, V.: p-Laplace equations with singular terms and p-superlinear perturbations. Lib. Math. (N.S.) **32**, 77–95 (2012)
3. Brézis, H.: Functional Analysis, Sobolev Spaces and Partial Differential Equations. Springer, New York (2011)
4. Canino, A., Degiovanni, M.: A variational approach to a class of singular semilinear elliptic equations. J. Convex Anal. **11**, 147–162 (2004)
5. Gasiński, L., Papageorgiou, N.S.: Nonlinear Analysis. Chapman & Hall/CRC, Boca Raton (2006)
6. Gasiński, L., Papageorgiou, N.S.: Nonlinear elliptic equations with singular terms and combined nonlinearities. Ann. Henri Poincaré **13**, 481–512 (2012)
7. Gasiński, L., Papageorgiou, N.S.: Bifurcation-type results for nonlinear parametric elliptic equations. Proc. R. Soc. Edinb. Sect. A **142**, 595–623 (2012)
8. Gasiński, L., Papageorgiou, N.S.: Multiplicity of positive solutions for eigenvalue problems of $(p, 2)$-equations. Bound. Value Probl. **152**, 17 (2012)
9. Gasiński, L., Papageorgiou, N.S.: A pair of positive solutions for the Dirichlet $p(z)$-Laplacian with concave and convex nonlinearities. J. Global Optim. **56**, 1347–1360 (2013)
10. Gasiński, L., Papageorgiou, N.S.: Dirichlet (p, q)-equations at resonance. Discrete Contin. Dyn. Syst. **34**, 2037–2060 (2014)
11. Gasiński, L., Papageorgiou, N.S.: Positive solutions for parametric equidiffusive p-Laplacian equations. Acta Math. Sci. Ser. B (Engl. Ed.) **34**, 610–618 (2014)
12. Gasiński, L., Papageorgiou, N.S.: Parametric p-Laplacian equations with superlinear reactions. Dyn. Syst. Appl. **24**, 523–558 (2015)
13. Gasiński, L., Papageorgiou, N.S.: Positive solutions for the generalized nonlinear logistic equations. Can. Math. Bull. **59**, 73–86 (2016)
14. Gasiński, L., Papageorgiou, N.S.: Positive, extremal and nodal solutions for nonlinear parametric problems. J. Convex Anal. **24**, 261–285 (2017)
15. Gasiński, L., Papageorgiou, N.S.: Positive solutions for the Neumann p-Laplacian with superdiffusive reaction. Bull. Malays. Math. Sci. Soc. **40**, 1711–1731 (2017)
16. Gasiński, L., Papageorgiou, N.S.: Multiplicity theorems for $(p, 2)$-equations. J. Nonlinear Convex Anal. **18**, 1297–1323 (2017)
17. Giacomoni, J., Saoudi, K.: $W_0^{1,p}$ versus C^1 local minimizers for a singular and critical functional. J. Math. Anal. Appl. **363**, 697–710 (2010)
18. Giacomoni, J., Schindler, I., Takáč, P.: Sobolev versus Hölder local minimizers and existence of multiple solutions for a singular quasilinear equation. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **6**, 117–158 (2007)
19. Gilbarg, D., Trudinger, N.S.: Elliptic Partial Differential Equations of Second Order. Springer, Berlin (2001)
20. Hirano, N., Saccon, C., Shioji, N.: Brezis–Nirenberg type theorems and multiplicity of positive solutions for a singular elliptic problem. J. Differ. Equ. **245**, 1997–2037 (2008)
21. Kyritsi, STh, Papageorgiou, N.S.: Pairs of positive solutions for singular p-Laplacian equations with a p-superlinear potential. Nonlinear Anal. **73**, 1136–1142 (2010)
22. Lazer, A.C., McKenna, P.J.: On a singular nonlinear elliptic boundary-value problem. Proc. Am. Math. Soc. **111**, 721–730 (1991)

23. Li, G., Yang, C.: The existence of a nontrivial solution to a nonlinear elliptic boundary value problem of p-Laplacian type without the Ambrosetti-Rabinowitz condition. Nonlinear Anal. **72**, 4602–4613 (2010)
24. Marano, S.A., Papageorgiou, N.S.: Positive solutions to a Dirichlet problem with p-Laplacian and concave-convex nonlinearity depending on a parameter. Commun. Pure Appl. Anal. **12**, 815–829 (2013)
25. Mohammed, A.: Positive solutions of the p-Laplace equation with singular nonlinearity. J. Math. Anal. Appl. **352**, 234–245 (2009)
26. Motreanu, D., Motreanu, V.V., Papageorgiou, N.S.: Topological and Variational Methods with Applications to Nonlinear Boundary Value Problems. Springer, New York (2014)
27. Papageorgiou, N.S., Rădulescu, V., Repovš, D.D.: Pairs of positive solutions for resonant singular equations with the p-Laplacian. Electron. J. Differ. Equ. **249**, 13 (2017)
28. Papageorgiou, N.S., Smyrlis, G.: Nonlinear elliptic equations with singular reaction. Osaka J. Math. **53**, 489–514 (2016)
29. Papageorgiou, N.S., Smyrlis, G.: A bifurcation-type theorem for singular nonlinear elliptic equations. Methods Appl. Anal. **22**, 147–170 (2015)
30. Perera, K., Zhang, Z.: Multiple positive solutions of singular p-Laplacian problems by variational methods. Bound. Value Probl. **2005**, 377–382 (2005)
31. Sun, Y., Wu, S., Long, Y.: Combined effects of singular and superlinear nonlinearities in some singular boundary value problems. J. Differ. Equ. **176**, 511–531 (2001)
32. Zhao, L., He, Y., Zhao, P.: The existence of three positive solutions of a singular p-Laplacian problem. Nonlinear Anal. **74**, 5745–5753 (2011)